THE DISPERSIVE WAVE DYNAMICS OF A TWO-PHASE FLOW RELAXATION MODEL

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Abstract. We consider a general Euler-type two-phase flow model with relaxation towards phase equilibrium. We provide a complete description of the transition between the wave dynamics of the homogeneous relaxation system and that of the local equilibrium approximation. This transitional wave dynamics is fully determined by two parameters; a dimensionless stiffness parameter and the ratio of the sound velocities in the stiff and non-stiff limits.

We prove the Galilean invariance of the transitional waves, and show that their stability criterion is precisely the sub-characteristic condition. We further prove a maximum principle in the transitional regime, similar in spirit to the subcharacteristic condition; the transitional wave speeds can never exceed the largest wave speed of the homogeneous relaxation system. Finally, we identify the existence of a *critical region* of wave numbers where the sonic waves completely disappear from the system. This region corresponds to the *casus irreducibilis* of the describing cubic polynomial.

Relaxation; sub-characteristic condition; phase transfer.

1. Introduction

Based on the classical approach of Baer-Nunziato[4], a common way of modeling non-equilibrium two-phase flows is through hyperbolic relaxation models[2, 8, 14, 18, 19]. Recently, there has been significant interest in models for *cavitation* where a metastable gas-liquid mixture is moving with a single velocity[17, 20, 24].

In this paper, we consider 1D models of this type where thermal and mechanical equilibrium are assumed. Such simplified models have applications to two-phase pipeline flow relevant for environmental engineering and the petroleum industry[11]. In particular, we look at models that can be written in the general form[7, 10]

$$\partial_t(\alpha_{\mathrm{g}}\rho_{\mathrm{g}}) + \partial_x(\alpha_{\mathrm{g}}\rho_{\mathrm{g}}u) = \frac{1}{\varepsilon}(\mu_\ell - \mu_{\mathrm{g}}) \tag{1a}$$

$$\partial_t(\alpha_\ell\rho_\ell) + \partial_x(\alpha_\ell\rho_\ell u) = -\frac{1}{\varepsilon}(\mu_\ell - \mu_{\mathrm{g}}) \tag{1b}$$

$$\partial_t(\rho u) + \partial_x(\rho u^2 + p) = 0 \tag{1c}$$

$$\partial_t E + \partial_x(u(E+p)) = 0. \tag{1d}$$

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Herein, α_i is the volume fraction of phase i, ρ_i is the density, and u is the common velocity. For both phases i, the common pressure p satisfies a state relation

$$p = p(\rho_i, e_i) \tag{2}$$

where e_i is the specific internal energy. Furthermore, E is the total energy of the mixture, given by

$$E = \rho_{\mathrm{g}}\alpha_{\mathrm{g}}e_{\mathrm{g}} + \rho_\ell\alpha_\ell e_\ell + \frac{1}{2}\rho u^2, \tag{3}$$

where

$$\rho = \rho_{\mathrm{g}}\alpha_{\mathrm{g}} + \rho_\ell\alpha_\ell \tag{4}$$

is the mixture density. Finally, μ_i is the chemical potential given by

$$\mu_i = e_i + \frac{p}{\rho_i} - Ts_i, \tag{5}$$

where T is the common temperature and s_i is the specific entropy of phase i.

In this paper, our focus is on purely mathematical analysis of this model. In particular, we aim to provide a complete as possible description of how the wave dynamics depend on the thermodynamic parameters and the strength of the relaxation term.

For general hyperbolic relaxation systems, the zero relaxation limit has been the topic of much study[9, 15, 16]. In the system (1a)–(1d), ε can be seen as a time-scale of the relaxation process. In the limit $\varepsilon \to \infty$, the phase compositions of the mixture are frozen and the model becomes a hyperbolic conservation law for the masses, the total momentum and the total energy. In the limit $\varepsilon \to 0$, the relaxation system is formally equivalent to the equilibrium system

$$\mu_{\mathrm{g}} = \mu_\ell \tag{6a}$$

$$\partial_t\rho + \partial_x(\rho u) = 0, \tag{6b}$$

$$\partial_t(\rho u) + \partial_x(\rho u^2 + p) = 0, \tag{6c}$$

$$\partial_t E + \partial_x(u(E+p)) = 0, \tag{6d}$$

where the two mass balance equations are replaced with a single conservation law for the total mass.

The wave dynamics of the limiting cases for ε is well understood. In their recent work, Flåtten and Lund[7, 10] analyzed how the characteristic velocities of two-phase relaxation models depend on different assumptions on chemical, thermal and mechanical equilibrium. They found that for models of the type (1a)–(1d), the wave speeds of the equilibrium model will always be lower than for the full model. This is known from the general theory of hyperbolic relaxation systems as the *subcharacteristic condition*, which may be derived from entropy considerations[5] and is closely related to the stability of the relaxation process[22].

However, in a practical application the relaxation parameter ε will take finite values depending on the rate of mass transfer modeled by e.g. statistical rate theory[21]. Since the speed of sound in a pipeline can play a crucial role in important transient events such as crack propagation and emergency depressurization, this warrants the study of the wave dynamics in the transitional regime for which ε takes finite values.

While hyperbolic relaxation systems have been an active field of research for decades, the development of a general theory for the wave-dynamics in this transitional regime has been limited. Recently, some of the present authors studied the transitional regime of general 2×2 relaxation systems[3]. In this case, the transitional wave speeds were found to fulfill a transitional sub-characteristic condition. Moreover, a critical transition point was found for which the wave dynamics abruptly change from being similar to the equilibrium system to behaving more like the frozen system.

The purpose of the present paper is to analyze the transitional wave dynamics of the 1D phase relaxation model (1a)–(1d), by studying the individual Fourier components following the approach of Aursand and Flåtten[3]. In particular, we will show how the wave dynamics are fully determined by only two dimensionless parameters; a stiffness parameter and the ratio of the sound velocities in the limiting equilibrium and frozen models. We demonstrate that the transitional waves are stable if and only if the subcharacteristic condition is satisfied.

Also, similarly to the observation made in the similar work[3], a critical transition region between the equilibrium dynamics and the frozen dynamics will be identified. In this region, the sonic waves lose their physical meaning and are replaced with an indeterminate wave moving with the fluid velocity. Mathematically, this interesting phenomenon corresponds exactly to the *casus irreducibilis* of the cubic polynomial whose roots describe the Fourier components.

This paper is organized as follows. In Section 2, we review the theory of plane wave solutions to linear relaxation systems. In Section 3, we present an explicit linearization of the two-phase flow relaxation system considered in this paper. In Section 4, we present the characteristic polynomial whose roots describe the velocities and amplifications of the transitional waves. Herein, we prove that these solutions satisfy the expected Galilean symmetry under a change of inertial reference frame.

These roots are analyzed in Section 5. In Sections 5.1–5.2, we verify that we recover the frozen and equilibrium waves in the stiff and non-stiff limits. In Section 5.3, we prove that the transitional waves are stable if and only if the subcharacteristic condition is satisfied. In Section 5.4, we prove a principle that carries the essence of the subcharacteristic condition over to the transitional regime; for all wave numbers, the maximum wave speed of the relaxation system can never exceed the maximum wave speed of the frozen (non-stiff) limit of the system. In Section 5.5, we provide fully general closed-form expressions for the velocities and amplifications of the transitional waves.

In Section 6, we provide some interpretations and illustrations of these analytical results. In particular, we emphasize the fact that a *critical region* of wave numbers appears if the ratio between the sound velocities in the stiff and non-stiff limits is sufficiently small. Through this critical region, corresponding to the *casus irreducibilis* of the describing cubic polynomial, all waves propagate with the fluid velocity v and a continuous labeling of the separate waves becomes impossible. Hence this region can be interpreted as a set of wave numbers where the relaxation system, in a very qualitative way, changes character from behaving more like the equilibrium system to acting more like the frozen system.

Finally, in Section 7, the main insights of the paper are summarized.

2. Linearized relaxation systems

The transitional dynamics can be studied through linear analysis. A hyperbolic relaxation system such as the phase relaxation model (1a)–(1d) can be written in the general form[5]

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \frac{1}{\varepsilon}\boldsymbol{Q}(\boldsymbol{U}), \tag{7}$$

where $\boldsymbol{U} = \boldsymbol{U}(x,t) \in G \subseteq \mathbb{R}^N$ for some state space G.

Now let $\hat{\boldsymbol{U}}$ be some constant equilibrium state, characterized by

$$\boldsymbol{Q}(\hat{\boldsymbol{U}}) = 0. \tag{8}$$

The relaxation system (7) linearized around $\hat{\boldsymbol{U}}$ can then be written as

$$\partial_t \boldsymbol{V} + \boldsymbol{A}\partial_x \boldsymbol{V} = \frac{1}{\varepsilon}\boldsymbol{R}\boldsymbol{V}, \tag{9}$$

where

$$\boldsymbol{V} = \boldsymbol{U} - \hat{\boldsymbol{U}}, \tag{10}$$

and the constant matrices

$$\boldsymbol{A} = \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{U}} \qquad \text{and} \qquad \boldsymbol{R} = \frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{U}} \tag{11}$$

are evaluated at the equilibrium state $\hat{\boldsymbol{U}}$.

Remark 1. *Note that this linearization is valid also in the more general case where $\varepsilon = \varepsilon(\boldsymbol{U})$, if it can be assumed that ε is differentiable at equilibrium. Define*

$$\hat{\varepsilon} = \varepsilon(\hat{\boldsymbol{U}}). \tag{12}$$

Then it follows from (8) *that*

$$\frac{1}{\hat{\epsilon}}\boldsymbol{R} = \frac{\partial}{\partial \boldsymbol{U}}\left(\frac{1}{\varepsilon}\boldsymbol{Q}(\boldsymbol{U})\right) = \boldsymbol{Q}(\hat{\boldsymbol{U}})\frac{\partial}{\partial \boldsymbol{U}}\left(\frac{1}{\varepsilon}\right) + \frac{1}{\hat{\varepsilon}}\frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{U}} = \frac{1}{\hat{\varepsilon}}\frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{U}}. \tag{13}$$

2.1. Plane-wave solutions.

For an initial condition $\boldsymbol{V}(x,0) \in L^2(\mathbb{R})$, there exists a unique solution to (9)[6]. If $\boldsymbol{V}(x,0) \in L^2(\mathbb{T})$ with $\mathbb{T} \subset \mathbb{R}$ and $\mathbb{T}$ is of length $T < \infty$, the solution can be written in the general form

$$\boldsymbol{V}(x,t) = \sum_k \boldsymbol{V}_k(x,t) = \sum_k \exp(\boldsymbol{H}(k)t)\exp(ikx)\hat{\boldsymbol{V}}(k), \tag{14}$$

where k is the wave number and $\boldsymbol{H}$ is a wave-number dependent matrix given by

$$\boldsymbol{H} = \frac{1}{\varepsilon}\boldsymbol{R} - ik\boldsymbol{A}. \tag{15}$$

We now assume that $\boldsymbol{H}$ is diagonalizable, i. e. it can be written in the form

$$\boldsymbol{H} = \boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^{-1}, \tag{16}$$

where $\boldsymbol{P}$ is its matrix of eigenvectors and $\boldsymbol{\Lambda}$ the diagonal matrix of eigenvalues. By using (16), we may then write the general solution (14) in terms of plane waves as

$$\boldsymbol{V}(x,t) = \sum_k \sum_{j=1}^{N} \bar{V}_j(k)\exp(ikx + \lambda_j t), \tag{17}$$

for some amplitudes $\bar{V}_j(k)$. Now to each eigenvalue λ_j of $\boldsymbol{H}(k)$ there is an associated plane wave with velocity

$$v_j(k) = -\frac{1}{k}\mathrm{Im}(\lambda_j) \tag{18}$$

and amplification factor

$$f_j(k) = \mathrm{Re}(\lambda_j), \tag{19}$$

as can be seen by writing (17) as

$$\boldsymbol{V}(x,t) = \sum_k \sum_{j=1}^{N} \bar{V}_j(k)\exp(f_j t)\exp\left(ik(x - v_j t)\right). \tag{20}$$

It now follows from (15) that $\boldsymbol{H}$ satisfies the symmetry

$$\boldsymbol{H}(k) = \overline{\boldsymbol{H}(-k)}. \tag{21}$$

Hence

$$\lambda_j(k) = \overline{\lambda_j(-k)} \tag{22}$$

and consequently

$$f_j(k) = f_j(-k), \tag{23}$$
$$v_j(k) = v_j(-k). \tag{24}$$

Hence we may with no loss of generality restrict the analysis of this paper to non-negative wave numbers, i. e. we assume

$$k \in [0, \infty). \tag{25}$$

3. The phase relaxation system

We now aim to derive an explicit expression for the matrix $\boldsymbol{H}$ corresponding to the phase relaxation system (1a)–(1d). The Jacobian matrix $\boldsymbol{A}$ for this system is given by[1, 8, 12]

$$\boldsymbol{A} = \begin{bmatrix} (1-Y)u & -Yu & Y & 0 \\ -(1-Y)u & Yu & 1-Y & 0 \\ a_{\mathrm{g}} - u^2 & a_\ell - u^2 & 2u - uP_\epsilon & P_\epsilon \\ u\left(a_{\mathrm{g}} - \frac{E+p}{\rho}\right) & u\left(a_\ell - \frac{E+p}{\rho}\right) & \frac{E+p}{\rho} - u^2 P_\epsilon & u\left(P_\epsilon + 1\right) \end{bmatrix}, \tag{26}$$

where

$$Y = \frac{\rho_{\mathrm{g}}\alpha_{\mathrm{g}}}{\rho}, \tag{27}$$

$$P_\epsilon = \frac{\rho\tilde{c}^2}{T}\frac{\zeta_{\mathrm{g}}C_{p,\mathrm{g}} + \zeta_\ell C_{p,\ell}}{C_{p,\mathrm{g}} + C_{p,\ell}} \tag{28}$$

and

$$a_i = \frac{\rho\tilde{c}^2}{\rho_i} + \left(\frac{1}{2}u^2 - e_i - \frac{p}{\rho_i}\right)P_\epsilon, \qquad i \in \{\mathrm{g}, \ell\}. \tag{29}$$

Herein, we have used the thermodynamic parameters[8]:

$$C_{p,i} = \rho_i \alpha_i c_{p,i} = \rho_i \alpha_i T \left(\frac{\partial s_i}{\partial T}\right)_p, \tag{30}$$

$$\zeta_i = \left(\frac{\partial T}{\partial p}\right)_{s_i} = -\frac{1}{\rho_i^2}\left(\frac{\partial \rho_i}{\partial s_i}\right)_p. \tag{31}$$

Furthermore, $\tilde{c}$ is the sound velocity corresponding to the limit $\varepsilon \to \infty$ in (1a)–(1d), and is given by[8]:

$$\tilde{c}^{-2} = \frac{\alpha_{\mathrm{g}}}{\rho_{\mathrm{g}} c_{\mathrm{g}}^2} + \frac{\alpha_\ell}{\rho_\ell c_\ell^2} + \frac{C_{p,\mathrm{g}} C_{p,\ell} (\zeta_{\mathrm{g}} - \zeta_\ell)^2}{T(C_{p,\mathrm{g}} + C_{p,\ell})}, \tag{32}$$

where

$$c_i^2 = \left(\frac{\partial p}{\partial \rho_i}\right)_{s_i} \tag{33}$$

are the one-phase sound velocities.

3.1. **The relaxation matrix.** In this section, we derive the linearized relaxation matrix $\boldsymbol{R}$, i. e. the Jacobian of the vector

$$\boldsymbol{Q} = \begin{bmatrix} \mu_\ell - \mu_{\mathrm{g}} \\ \mu_{\mathrm{g}} - \mu_\ell \\ 0 \\ 0 \end{bmatrix} \tag{34}$$

evaluated at equilibrium. We start with establishing a useful differential.

Lemma 1. *The differential for the chemical potential difference can be written as*

$$\mathrm{d}(\mu_{\mathrm{g}} - \mu_\ell) = \left(\frac{\rho \tilde{c}^2 \beta}{\rho_{\mathrm{g}}} - h_{\mathrm{g}} \mathcal{M}_\epsilon\right) \mathrm{d}(\rho_{\mathrm{g}} \alpha_{\mathrm{g}}) + \left(\frac{\rho \tilde{c}^2 \beta}{\rho_\ell} - h_\ell \mathcal{M}_\epsilon\right) \mathrm{d}(\rho_\ell \alpha_\ell) + \mathcal{M}_\epsilon \, \mathrm{d}\epsilon \tag{35}$$

where

$$\Theta = \frac{\zeta_{\mathrm{g}} C_{p,\mathrm{g}} + \zeta_\ell C_{p,\ell}}{C_{p,\mathrm{g}} + C_{p,\ell}}, \tag{36}$$

$$\beta = \frac{1}{\rho_{\mathrm{g}}} - \frac{1}{\rho_\ell} + \Theta(s_\ell - s_{\mathrm{g}}), \tag{37}$$

$$\mathcal{M}_\epsilon = \mathcal{P}_\epsilon \beta + \frac{s_\ell - s_{\mathrm{g}}}{C_{p,\mathrm{g}} + C_{p,\ell}}, \tag{38}$$

$$\epsilon = \rho_{\mathrm{g}} \alpha_{\mathrm{g}} e_{\mathrm{g}} + \rho_\ell \alpha_\ell e_\ell, \tag{39}$$

$$h_k = e_k + \frac{p}{\rho_k}, \quad k \in \{\mathrm{g}, \ell\}. \tag{40}$$

Proof. From the Legendre transform on the fundamental differential

$$\mathrm{d}e_k = T \, \mathrm{d}s_k + \frac{p}{\rho_k^2} \, \mathrm{d}\rho_k \tag{41}$$

we obtain

$$\mathrm{d}(\mu_{\mathrm{g}} - \mu_\ell) = \left(\frac{1}{\rho_{\mathrm{g}}} - \frac{1}{\rho_\ell}\right) \mathrm{d}p + (s_\ell - s_{\mathrm{g}}) \, \mathrm{d}T. \tag{42}$$

Furthermore, the following relation was derived in Flåtten et al.[8]:

$$\frac{\mathrm{d}(\rho_{\mathrm{g}}\alpha_{\mathrm{g}})}{\rho_{\mathrm{g}}}+\frac{\mathrm{d}(\rho_\ell\alpha_\ell)}{\rho_\ell}=\left(\frac{\alpha_{\mathrm{g}}}{\rho_{\mathrm{g}}c_{\mathrm{g}}^2}+\frac{\alpha_\ell}{\rho_\ell c_\ell^2}+\frac{\zeta_{\mathrm{g}}^2C_{p,\mathrm{g}}+\zeta_\ell^2C_{p,\ell}}{T}\right)\mathrm{d}p-\frac{\zeta_{\mathrm{g}}C_{p,\mathrm{g}}+\zeta_\ell C_{p,\ell}}{T}\,\mathrm{d}T, \tag{43}$$

and from Morin et al.[12] we have:

$$\mathrm{d}p=\left(\frac{\rho\tilde{c}^2}{\rho_{\mathrm{g}}}-\mathcal{P}_\epsilon h_{\mathrm{g}}\right)\mathrm{d}(\rho_{\mathrm{g}}\alpha_{\mathrm{g}})+\left(\frac{\rho\tilde{c}^2}{\rho_\ell}-\mathcal{P}_\epsilon h_\ell\right)\mathrm{d}(\rho_\ell\alpha_\ell)+\mathcal{P}_\epsilon\,\mathrm{d}\epsilon. \tag{44}$$

The result now follows from substituting (43) and (44) in (42).

□

This result gives us an explicit formulation of the relaxation matrix.

Proposition 1. *In the context of* (11)*, the relaxation matrix* $\boldsymbol{R}$ *corresponding to the system* (1a)–(1d) *is given by*

$$\boldsymbol{R}=\begin{bmatrix}-R_1 & -R_2 & -R_3 & -R_4\\ R_1 & R_2 & R_3 & R_4\\ 0&0&0&0\\ 0&0&0&0\end{bmatrix}, \tag{45}$$

where

$$R_1=\frac{\rho\tilde{c}^2\beta}{\rho_{\mathrm{g}}}+\left(\frac{1}{2}u^2-h_{\mathrm{g}}\right)\mathcal{M}_\epsilon, \tag{46}$$

$$R_2=\frac{\rho\tilde{c}^2\beta}{\rho_\ell}+\left(\frac{1}{2}u^2-h_\ell\right)\mathcal{M}_\epsilon, \tag{47}$$

$$R_3=-u\mathcal{M}_\epsilon, \tag{48}$$

$$R_4=\mathcal{M}_\epsilon. \tag{49}$$

Proof. With

$$\boldsymbol{U}=\begin{bmatrix}U_1\\U_2\\U_3\\U_4\end{bmatrix}=\begin{bmatrix}\rho_{\mathrm{g}}\alpha_{\mathrm{g}}\\ \rho_\ell\alpha_\ell\\ \rho u\\ E\end{bmatrix}, \tag{50}$$

we have the relations

$$\mathrm{d}(\rho_{\mathrm{g}}\alpha_{\mathrm{g}})=\mathrm{d}U_1, \tag{51}$$

$$\mathrm{d}(\rho_\ell\alpha_\ell)=\mathrm{d}U_2, \tag{52}$$

$$\mathrm{d}\epsilon=\frac{1}{2}u^2\,(\mathrm{d}U_1+\mathrm{d}U_2)-u\,\mathrm{d}U_3+\mathrm{d}U_4. \tag{53}$$

The result now follows from substituting (51)–(53) into (35).

□

4. The characteristic polynomial

From Proposition 1 and (26), we may now directly construct the wave-number dependent matrix $\boldsymbol{H}$ as defined by (15). In equilibrium, the chemical potentials (5) are equal, so we may write

$$s_{\mathrm{g}}-s_\ell=\frac{h_{\mathrm{g}}-h_\ell}{T}. \tag{54}$$

Then a direct calculation of the characteristic polynomial of $\boldsymbol{H}_k$ gives:

$$\begin{aligned}\lambda^4 + \left(\frac{\gamma\tilde{c}^2}{\varepsilon\hat{c}^2} + 4iku\right)\lambda^3 + \left(k^2\tilde{c}^2 - 6k^2u^2 + 3iku\frac{\gamma\tilde{c}^2}{\varepsilon\hat{c}}\right)\lambda^2 \\ + \left(-3\frac{k^2u^2}{\varepsilon}\gamma\frac{\tilde{c}^2}{\hat{c}^2} + \frac{k^2}{\varepsilon}\tilde{c}^2\gamma - 4ik^3u^3 + 2ik^3u\tilde{c}^2\right)\lambda \\ +k^4u^2(u^2-\tilde{c}^2) - i\frac{k^3u^3}{\varepsilon}\frac{\tilde{c}^2}{\hat{c}^2}\gamma + i\frac{k^3u}{\varepsilon}\tilde{c}^2\gamma = 0,\end{aligned} \tag{55}$$

where the parameter $\hat{c}$ corresponds to the mixture sound velocity of the equilibrium system (6a)–(6b). It is given by[7, 13, 19]

$$\hat{c}^{-2} = \rho\left(\frac{\alpha_{\mathrm{g}}}{\rho_{\mathrm{g}}c_{\mathrm{g}}^2} + \frac{\alpha_\ell}{\rho_\ell c_\ell^2} + T\left(C_{p,\mathrm{g}}\left(\frac{\zeta_{\mathrm{g}}}{T} + W\right)^2 + C_{p,\ell}\left(\frac{\zeta_\ell}{T} + W\right)^2\right)\right), \tag{56}$$

where

$$W = \frac{1}{\rho_{\mathrm{g}}\rho_\ell}\frac{\rho_{\mathrm{g}} - \rho_\ell}{h_{\mathrm{g}} - h_\ell}. \tag{57}$$

We have also introduced the shorthand

$$\gamma = \frac{(h_{\mathrm{g}} - h_\ell)^2}{T(C_{p,\mathrm{g}} + C_{p,\ell})}. \tag{58}$$

Note that the sound velocities satisfy the *subcharacteristic condition*[5, 9]

$$\hat{c} \le \tilde{c} \tag{59}$$

subject only to fundamental thermodynamic stability constraints[7].

4.1. **Galilean invariance.** We can demonstrate that the roots of the characteristic polynomial possess the expected Galilean symmetry, i.e. they are invariant under a change of inertial reference frame.

By introducing the dimensionless parameters

$$\varphi = k\varepsilon\frac{\hat{c}^2}{\gamma\tilde{c}}, \qquad y = \frac{\lambda}{k\tilde{c}}, \qquad r = \frac{\hat{c}}{\tilde{c}}, \tag{60}$$

we may transform the polynomial (55) to

$$\begin{aligned}\varphi\left(y + i\frac{u}{\tilde{c}}\right)^2\left(y + i\left(\frac{u}{\tilde{c}} + 1\right)\right)\left(y + i\left(\frac{u}{\tilde{c}} - 1\right)\right) \\ + \left(y + i\frac{u}{\tilde{c}}\right)\left(y + i\left(\frac{u}{\tilde{c}} + r\right)\right)\left(y + i\left(\frac{u}{\tilde{c}} - r\right)\right) = 0.\end{aligned} \tag{61}$$

From (61) it is clear that the change of variables

$$z = y + i\frac{u}{\tilde{c}} \tag{62}$$

yields the polynomial

$$\varphi z^2(z+i)(z-i) + z(z+ir)(z-ir) = 0, \tag{63}$$

which is indeed independent of the velocity u.

It is worth noting that the simple polynomial (63) now gives a *complete* description of the wave-number dependent velocities and amplifications for a general relaxation model in the form (1a)–(1d). Remarkably, this dynamics is uniquely determined only by the parameters φ and r as defined by (60). And, as stated by

Remark 1, the model presented in Section 3 is sufficiently general to represent *any* such model where the relaxation term satisfies only two natural properties:

- The mass transfer term disappears in equilibrium where $\mu_{\mathrm{g}} = \mu_\ell$.
- The relaxation parameter ε is differentiable across the equilibrium state.

In the following sections, we will study the transitional wave dynamics in full detail.

5. Transitional wave dynamics

We first consider the wave dynamics in the limiting cases $\varepsilon \to \infty$ and $\varepsilon \to 0$ respectively, corresponding to the frozen limit of (1a)–(1d) and the equilibrium limit (6a)–(6b).

5.1. **The frozen limit.**

In the limit of infinite relaxation time, corresponding to $\varphi \to \infty$, the polynomial (63) reduces to

$$z^2(z+i)(z-i) = 0 \tag{64}$$

with the following 4 roots:

- $z = 0$ with multiplicity 2, corresponding to two waves with velocity u and zero amplification.
- $z = \pm i$, corresponding to two waves with velocities $u \pm \tilde{c}$ and zero amplification.

This is precisely the wave structure of the frozen model (1a)–(1d)[7, 8]. Herein, the two waves of velocity u physically represent entropy and mass fraction waves, whereas the waves of velocity $u \pm \tilde{c}$ are sonic waves. Note that from (60), there are three distinct ways this frozen limit can be physically realized:

- the limit of *infinite wave number* $k \to \infty$;
- the limit of *infinite relaxation time* $\varepsilon \to \infty$;
- the limit of *equal phasic enthalpies* giving $\gamma \to 0$ in (58).

5.2. **The stiff limit.**

In the limit of zero relaxation time, corresponding to $\varphi \to 0$, the polynomial (63) reduces to

$$z(z+ir)(z-ir) = 0. \tag{65}$$

This has the following 3 roots:

- $z = 0$, corresponding to a wave with velocity u and zero amplification.
- $z = \pm ir$, corresponding to two waves with velocities $u \pm \hat{c}$ and zero amplification.

Hence we recover the wave structure of the equilibrium model (6a)–(6d)[7, 19]. Herein, the wave of velocity u physically represents an entropy wave, whereas the waves of velocity $u \pm \hat{c}$ are sonic waves. From (60), there are now four distinct ways this stiff limit can be physically realized:

- the limit of *zero wave number* $k \to 0$;
- the limit of *zero relaxation time* $\varepsilon \to 0$;
- the limit of *zero temperature* giving $\gamma \to \infty$ in (58);
- the limit of *zero heat capacities* giving $\gamma \to \infty$ in (58).

5.3. **Transitional stability.** In this section, we establish a direct connection between the subcharacteristic condition (59) and the stability of the transitional waves as described by (63). We first observe that for all φ, (63) has a trivial root $z = 0$ corresponding to the entropy wave. Hence this wave may be fully described as follows; it will propagate with velocity u, and no amplification, independent of the wave number and relaxation time.

Eliminating this trivial root from (63), we obtain

$$\varphi z(z^2+1) + (z^2 + r^2) = 0. \tag{66}$$

For the plane wave solutions (14) to be stable, the amplification factors (19) must be non-positive. This is equivalent to the requirement that the roots z_i of (66) satisfy $\operatorname{Re} z_i \leq 0$.

Proposition 2 (Linear Stability)**.** *Let $\varphi \in (0,\infty)$. The real part of the roots of the polynomial* (66) *are nonpositive if and only if the subcharacteristic condition* (59) *is satisfied, i.e.*

$$0 \leq r \leq 1. \tag{67}$$

Proof. This general result follows as a special case of the Routh–Hurwitz theorem. For the purpose of illustration, we also state below a simple direct proof for the case where two of the roots are complex conjugate. A direct proof for the remaining case of distinct real roots may be derived along the same lines.

The polynomial (66) can be written in the form

$$\begin{aligned} z^3 + \frac{1}{\varphi}z^2 + z + \frac{r^2}{\varphi} &= (z-z_2)(z-z_1)(z-z_0) \\ &= z^3 - (z_2+z_1+z_0)z^2 + (z_2z_1 + z_1z_0 + z_0z_2)z - z_2z_1z_0 = 0. \end{aligned} \tag{68}$$

We can, without loss of generality, assume z_2 is real while z_1 and z_0 are conjugate roots ($z_0z_1 = |z_0|^2$). Assume first that the polynomial is stable, i.e. $\mathrm{Re} z_i \leq 0$. We can then write

$$\begin{aligned} \frac{1}{\varphi}\cdot 1 &= -(z_2+z_1+z_0)(z_2z_1+z_1z_0+z_0z_2) \\ &= \frac{r^2}{\varphi} - z_2^2(z_0+z_1) - |z_0|^2(z_0+z_1) - z_2(z_0+z_1)^2 \geq \frac{r^2}{\varphi}. \end{aligned} \tag{69}$$

by using (68).

For the converse statement, assume $0 \leq r \leq 1$. From the positivity of φ we have

$$\frac{r^2}{\varphi} = -z_2z_1z_0 = -z_2|z_0|^2 \geq 0, \tag{70}$$

which directly implies $z_2 = \mathrm{Re} z_2 \leq 0$. It remains to show that $z_0 + z_1 \leq 0$. Using the assumption and (68) we can write, after multiplying by -1,

$$z_0 + z_1 + z_2 \leq z_2|z_0|^2. \tag{71}$$

Furthermore, we have

$$1 = z_2z_1 + z_1z_0 + z_0z_2 = |z_0|^2 + z_2(z_0+z_1), \tag{72}$$

so we can then infer

$$z_0 + z_1 \leq z_2(|z_0|^2 - 1) = -z_2^2(z_0+z_1), \tag{73}$$

which completes the proof. ◻

Proposition 2 shows that for the phase relaxation model the notion of linear stability and the subcharacteristic condition are in fact equivalent.

5.4. **A maximum principle.** For general 2×2 relaxation systems, it was shown by Aursand and Flåtten[3] that the transitional wave velocities always satisfy a *monotonicity principle*; the velocities $v_j(k)$ are in this case monotonic functions of the stiffness parameter φ.

As will be demonstrated in Section 6, this monotonicity principle does not carry over to the relaxation model considered in this paper. However, a weaker constraint on the velocities may be derived.

Proposition 3 (Maximum principle)**.** *Let $\varphi \in (0, \infty)$. Assume that the subcharacteristic condition is satisfied, i.e. $r^2 \leq 1$ in the context of* (66)*. Then the imaginary parts of the roots in* (66) *satisfy* $|\mathrm{Im}(z)| \leq 1$.

Proof. Let $z = a + ib$, where $b = \mathrm{Im}(z)$ and $a = \mathrm{Re}(z)$. Then (66) is equal to

$$\varphi\left(a^3 - 3ab^2 + a\right) + a^2 - b^2 + r^2 + i\left[\varphi\left(3a^2 b - b^3 + b\right) + 2ab\right] = 0. \tag{74}$$

Both the real and the imaginary part of (74) have to be equal to zero, giving us, for the imaginary part,

$$b^2 - 1 = \frac{2a}{\varphi} + 3a^2. \tag{75}$$

It follows from Proposition 2 that $a \leq 0$ when $r^2 \leq 1$. If $a = 0$, then $b^2 = 1$. For the case $a < 0$, let us assume that $|b| > 1$. Then $b^2 - 1 > 0$ and, dividing (75) by a, we are left with the inequality

$$a < -\frac{2}{3\varphi}, \tag{76}$$

since $a < 0$. Multiplying the imaginary part of (74) with a/b and subtracting it from 3 times the real part, we get

$$-8\varphi ab^2 + 2\varphi a + a^2 - 3b^2 + 3r^2 = 0. \tag{77}$$

By subtracting 1/3 times (75) from (77), we then have

$$a\left(-8\varphi b^2 + 2\varphi - \frac{2}{3\varphi}\right) = \frac{8}{3}b^2 - 3r^2 + \frac{1}{3}. \tag{78}$$

With a satisfying (76), we get

$$-\frac{2}{3\varphi}\left(-8\varphi b^2 + 2\varphi - \frac{2}{3\varphi}\right) < \frac{8}{3}b^2 - 3r^2 + \frac{1}{3}, \tag{79}$$

which results in the inequality

$$-r^2 > \frac{8b^2}{9} - \frac{5}{9} + \frac{4}{27\varphi^2}. \tag{80}$$

With $|b| > 1$ the right hand side in (80) is positive, but the left hand side is negative, which is a contradiction. Thus $|b| \leq 1$.

□

Physically, this result may be interpreted as a causality principle; the relaxation terms cannot be used to increase the velocity of information propagation in a stable system. Note that for our particular two-phase flow relaxation model, Proposition 3 is equivalent to the statement

$$u - \tilde{c} \le v_j(k) \le u + \tilde{c} \quad \forall j, k. \tag{81}$$

5.5. **Analytical solutions.** Given that (66) is a cubic polynomial, it can be solved exactly. In this section, we provide explicit expressions for the roots and provide some interpretations of the results.

The discriminant of (66) is

$$\Delta = -\left(\frac{2}{3}(3\varphi^2 - 1)\right)^2 - \left(r^2 - \frac{1}{9}\right)(27\varphi^2 r^2 - 15\varphi^2 + 4). \tag{82}$$

It will now be convenient to introduce the auxiliary variables

$$s = r^2 - \frac{1}{9}, \tag{83}$$

$$\omega = 3\varphi^2 - 1, \tag{84}$$

enabling us to write (82) in the simple form

$$\Delta = -\left(\frac{2}{3}\omega - 3s\right)^2 - 9s^2\omega. \tag{85}$$

Now the nature of the roots of (66) are determined by the sign of the discriminant as follows:

D1: $\Delta < 0$: The equation (66) has one real root and two complex conjugate roots.

D2: $\Delta \ge 0$: The equation (66) has three real roots (casus irreducibilis).

Note that in the context of the transformation (62), a real root corresponds to a wave of velocity u. Hence the situation D2 (casus irreducibilis) corresponds to a *critical region* in wave numbers and relaxation times where the transitional sound velocities become zero. In fact, as will be described in the following, no continuous labeling of the waves as "sonic" or "mass fraction" can be made through this critical region.

5.5.1. *Critical region.* We now define the critical region $\mathcal{C}(r)$ as

$$\mathcal{C}(r) := \{\varphi \in (0, \infty) : \Delta(\varphi, r) \ge 0\}. \tag{86}$$

We may then state the following proposition.

Proposition 4. *The equation* (66) *has three real roots if and only if*

$$\varphi \in \mathcal{C}(r), \tag{87}$$

where

$$\mathcal{C}(r) = \begin{cases} \varnothing & \text{if } r > \frac{1}{3}, \\ [\varphi_c^-, \varphi_c^+] & \text{if } r \in [0, 1/3]. \end{cases} \tag{88}$$

Herein, $\varphi_{\mathrm{c}}^{\pm}$ *are given by:*

$$\varphi_{\mathrm{c}}^{-} = \frac{1}{2\sqrt{2}}\sqrt{9(2r^2 - 3r^4) + 1 - \sqrt{(r^2-1)(9r^2-1)^3}}, \tag{89}$$

$$\varphi_{\mathrm{c}}^{+} = \frac{1}{2\sqrt{2}}\sqrt{9(2r^2 - 3r^4) + 1 + \sqrt{(r^2-1)(9r^2-1)^3}}. \tag{90}$$

Proof. The discriminant (85) changes sign when the parameter ω satisfies

$$\omega_{\mathrm{crit}} = s\left(\frac{9}{2} - \frac{81}{8}s \pm \frac{27}{8}\sqrt{9s^2 - 8s}\right), \tag{91}$$

which corresponds to real-valued φ only if $r^2 \leq 1/9$. In this case, the transformation (84) gives (89)–(90) which are unconditionally positive. □

We are now in the position to state explicit formulae for the roots inside and outside of the critical region.

5.5.2. *One real and two complex solutions.* We now assume that $\Delta < 0$, i. e. $\varphi \notin \mathcal{C}(r)$. We now define

$$Q = \frac{1}{2}\left(8\omega - 108s\varphi^2 + 12\varphi\sqrt{-3\Delta}\right)^{1/3}. \tag{92}$$

Then the real solution may be written as

$$z_1 = \frac{1}{3\varphi}\left(Q - \frac{\omega}{Q} - 1\right), \tag{93}$$

whereas the complex solutions are given by

$$z_{2,3} = -\frac{1}{6\varphi}\left(Q - \frac{\omega}{Q} + 2 \pm \sqrt{3}i\left(Q + \frac{\omega}{Q}\right)\right). \tag{94}$$

5.5.3. *Casus irreducibilis.* We now consider the case $\Delta \geq 0$, i. e. $\varphi \in \mathcal{C}(r)$. We may put the polynomial (66) in the reduced form

$$t^3 + pt + q = 0, \tag{95}$$

where

$$t = z + \frac{1}{3\varphi}, \tag{96}$$

$$p = \frac{3\varphi^2 - 1}{3\varphi^2} = \frac{\omega}{3\varphi^2}, \tag{97}$$

$$q = \frac{2 - 9\varphi^2 + 27r^2\varphi^2}{27\varphi^3} = \frac{s}{\varphi} - \frac{2\omega}{27\varphi^3}. \tag{98}$$

Now the roots are given by $(k = 0, 1, 2)$,

$$t_k = \frac{2}{3\varphi\zeta}\cos\left(\frac{1}{3}\arccos\left(\frac{27s\varphi^2\zeta^3}{2} + \zeta\right) + \frac{\pi(2k-1)}{3}\right) \tag{99}$$

where

$$\zeta = \frac{1}{\sqrt{-\omega}}. \tag{100}$$

This yields

$$z_k = \frac{1}{3\varphi}\left(\frac{2}{\zeta}\cos\left(\frac{1}{3}\arccos\left(\frac{27s\varphi^2\zeta^3}{2} + \zeta\right) + \frac{\pi(2k-1)}{3}\right) - 1\right). \tag{101}$$

6. Illustrations and discussion

The analytical expressions derived in the previous section are too intangible to provide much in terms of direct insights into the structure of the transitional waves. In this section, we will illuminate this structure through a graphical investigation of the expressions (93)–(94) and (101).

As the roots are continuous functions of the stiffness parameter φ, one would expect that we should be able to identify sonic waves $P^{\pm}$ and a mass fraction wave Y that are continuously transformed between the frozen and stiff limits. In fact, this naïve assumption breaks down in the critical region; herein, the waves "mix" and no continuous labeling is possible. Similarly to the critical *point* found for the the general 2×2 system[3], we may interpret the critical *region* as a well-defined "transitional regime" where the system changes character from being equilibrium-like to behaving more like the frozen system. This interpretation will be made precise in the following.

6.1. **Frozen-like and equilibrium-like waves.**

For the purposes of the ensuing discussion, it will be convenient to introduce the following labeling of the waves, as given by the complex roots of the equation (63).

- *Frozen-like sonic waves* $P_{\mathrm{f}}^{\pm}$, which continuously transform into the frozen sonic waves as the limit $\varphi \to \infty$ is approached.
- *Equilibrium-like sonic waves* $P_{\mathrm{e}}^{\pm}$, which continuously transform into the equilibrium sonic waves as the limit $\varphi \to 0$ is approached.
- *A frozen-like mass fraction wave* Y_{f}, which continuously transforms into the frozen mass fraction wave as the limit $\varphi \to \infty$ is approached.
- *An equilibrium-like mass fraction wave* Y_{e}, whose amplification factor tends to $-\infty$ as the limit $\varphi \to \infty$ is approached; i. e. the wave is fully damped and completely disappears from the system.
- *An indeterminate wave* X which exists only in the critical region. It cannot be naturally interpreted as neither a mass fraction nor a sonic wave, but instead serves to connect the equilibrium-like and frozen-like versions of these waves.
- *An entropy wave* S, corresponding to the trivial root $z = 0$ in (63). The dispersive dynamics of this wave is independent of φ; the velocity of propagation is u and the amplification factor is 0 for all values of φ.

We now present the corresponding analytical expressions for each of these waves in turn, disregarding the trivial entropy wave. We consider first the case $r > 1/3$, where there is no critical region $\mathcal{C}(r)$.

6.1.1. *Smooth transitional dynamics.*

Consider now the case $r > 1/3$. In this case, the transition between the frozen and equilibrium limits is smooth, and the equilibrium-like and frozen-like waves are identical. We define the following labeling of the solutions (93)–(94):

- Mass fraction wave:

$$\operatorname{Im}(Y_{\mathrm{f}}) = \operatorname{Im}(Y_{\mathrm{e}}) = 0, \tag{102}$$

$$\operatorname{Re}(Y_{\mathrm{f}}) = \operatorname{Re}(Y_{\mathrm{e}}) = \frac{1}{3\varphi}\left(Q - \frac{\omega}{Q} - 1\right). \tag{103}$$

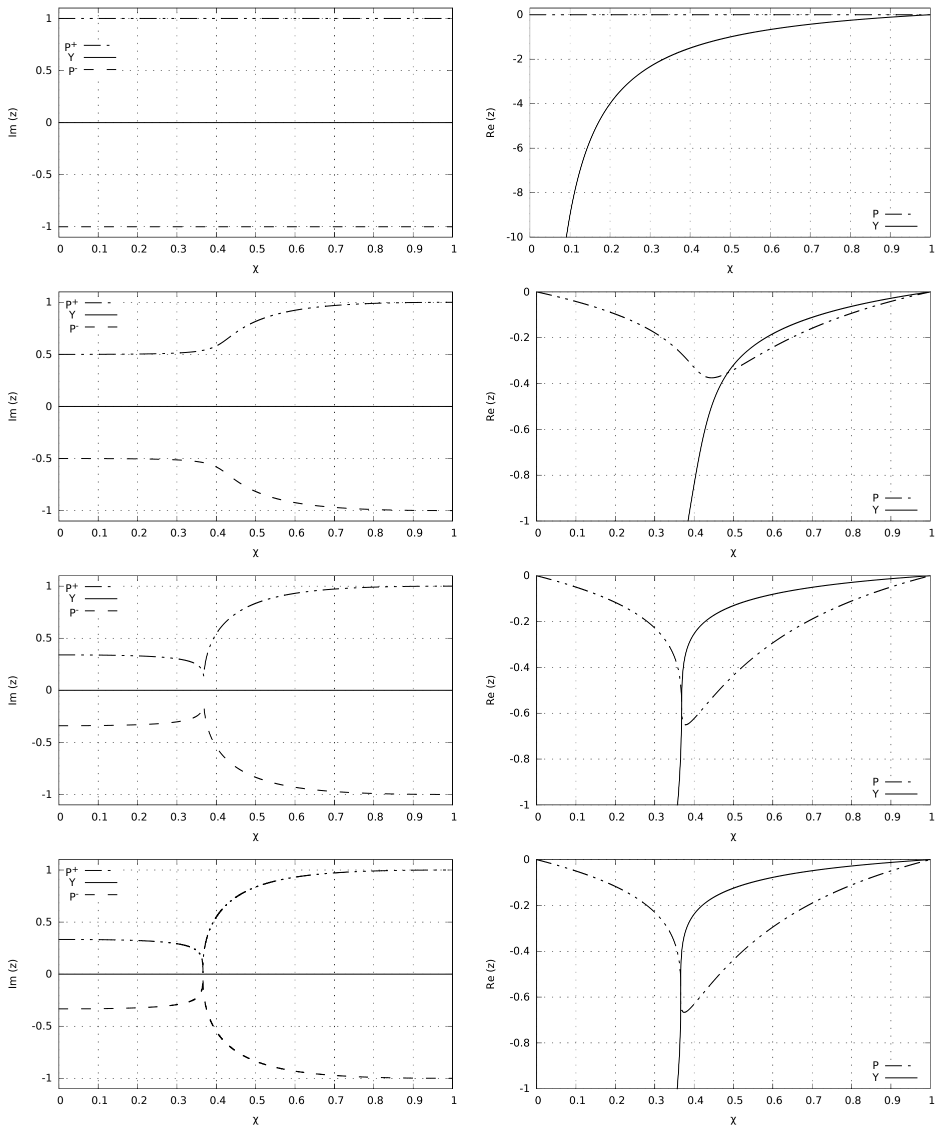

FIGURE 1. Transitional wave properties. Left: $\mathrm{Im}(z)$ (velocities). Right: $\mathrm{Re}(z)$ (amplifications). Top to bottom: $r = 1.0$, $r = 0.5$, $r = 0.34$, $r = 1/3$.

- Sonic waves:

$$\mathrm{Re}\left(P_{\mathrm{f}}^{\pm}\right) = \mathrm{Re}\left(P_{\mathrm{e}}^{\pm}\right) = -\frac{1}{6\varphi}\left(Q - \frac{\omega}{Q} + 2\right), \tag{104}$$

$$\mathrm{Im}\left(P_{\mathrm{f}}^{\pm}\right) = \mathrm{Im}\left(P_{\mathrm{e}}^{\pm}\right) = \pm\frac{1}{2\sqrt{3}\varphi}\left(Q + \frac{\omega}{Q}\right). \tag{105}$$

The amplifications and velocities of these waves are plotted in Figure 1, for different values of the parameter $r \in [1/3, 1]$. Herein, the variables are plotted as functions of a rescaled stiffness parameter χ, given by:

$$\chi(\varphi) = \frac{\varphi}{\varphi + 1}, \tag{106}$$

in order to limit the plotting domain to $[0, 1)$.

For $r = 1/3$, a critical phenomenon emerges; at the point

$$\varphi_\mathrm{c} = \sqrt{\frac{1}{3}}, \tag{107}$$

all the roots coincide, and the eigenspace of $\boldsymbol{H}$ as given by (15) degenerates. Hence the assumption (16) leading to the plane wave solutions (17) breaks down at this point. The solutions corresponding to the sonic velocities abruptly become zero, meaning that the sound waves in this limit propagate with the fluid velocity in the Eulerian frame. Also, around this point, the dampening of the sound waves is at its highest, and the dampening of the mass fraction wave (which disappears in the equilibrium system) increases sharply. Hence we can justify the statement that φ_c naturally divides the range of φ into an *equilibrium-like* and *frozen-like* region.

Decreasing the parameter r further causes this critical point φ_c to expand into a *critical region*. In this region, all the roots are distinct and the plane wave solutions (17) exist - but none of the solutions have a physical interpretation corresponding to sonic waves. This phenomenon will be investigated in the next section.

6.1.2. *Transition through critical region.* We consider now the case $r \in [0, 1/3]$, i. e. there exists a critical region and we assume that the subcharacteristic condition $r \leq 1$ is satisfied (as indeed has been proved for our particular model[7]). We define the following labeling of the solutions (93)–(94) and (101):

- Mass fraction wave:

$$\mathrm{Im}\,(Y_\mathrm{f}) = 0 \quad \text{for} \quad \varphi \geq \varphi_\mathrm{c}^-, \tag{108}$$

$$\mathrm{Im}\,(Y_\mathrm{e}) = 0 \quad \text{for} \quad \varphi \leq \varphi_\mathrm{c}^+, \tag{109}$$

$$\mathrm{Re}\,(Y_\mathrm{f}) = \begin{cases} \frac{1}{3\varphi}\left(Q - \frac{\omega}{Q} - 1\right) & \text{for} \quad \varphi > \varphi_\mathrm{c}^+ \\ \frac{1}{3\varphi}\left(\frac{2}{\zeta}\cos\left(\frac{1}{3}\arccos\left(\frac{27s\varphi^2\zeta^3}{2} + \zeta\right) - \frac{\pi}{3}\right) - 1\right) & \text{for} \quad \varphi_\mathrm{c}^- \leq \varphi \leq \varphi_\mathrm{c}^+, \end{cases} \tag{110}$$

$$\mathrm{Re}\,(Y_\mathrm{e}) = \begin{cases} \frac{1}{3\varphi}\left(Q - \frac{\omega}{Q} - 1\right) & \text{for} \quad \varphi < \varphi_\mathrm{c}^- \\ \frac{1}{3\varphi}\left(\frac{2}{\zeta}\cos\left(\frac{1}{3}\arccos\left(\frac{27s\varphi^2\zeta^3}{2} + \zeta\right) + \pi\right) - 1\right) & \text{for} \quad \varphi_\mathrm{c}^- \leq \varphi \leq \varphi_\mathrm{c}^+. \end{cases} \tag{111}$$

- Sonic waves:

$$\operatorname{Re}\left(P_{\mathrm{f}}^{\pm}\right) = -\frac{1}{6\varphi}\left(Q - \frac{\omega}{Q} + 2\right) \quad \text{for} \quad \varphi > \varphi_{\mathrm{c}}^{+}, \tag{112}$$

$$\operatorname{Im}\left(P_{\mathrm{f}}^{\pm}\right) = \pm\frac{1}{2\sqrt{3}\varphi}\left(Q + \frac{\omega}{Q}\right) \quad \text{for} \quad \varphi > \varphi_{\mathrm{c}}^{+}, \tag{113}$$

$$\operatorname{Re}\left(P_{\mathrm{e}}^{\pm}\right) = -\frac{1}{6\varphi}\left(Q - \frac{\omega}{Q} + 2\right) \quad \text{for} \quad \varphi < \varphi_{\mathrm{c}}^{-}, \tag{114}$$

$$\operatorname{Im}\left(P_{\mathrm{e}}^{\pm}\right) = \pm\frac{1}{2\sqrt{3}\varphi}\left(Q + \frac{\omega}{Q}\right) \quad \text{for} \quad \varphi < \varphi_{\mathrm{c}}^{-}. \tag{115}$$

- Indeterminate wave:

$$\operatorname{Im}(X) = 0 \quad \text{for} \quad \varphi \in [\varphi_{\mathrm{c}}^{-}, \varphi_{\mathrm{c}}^{+}], \tag{116}$$

$$\operatorname{Re}(X) = \frac{1}{3\varphi}\left(\frac{2}{\zeta}\cos\left(\frac{1}{3}\arccos\left(\frac{27s\varphi^2\zeta^3}{2} + \zeta\right) + \frac{\pi}{3}\right) - 1\right) \quad \text{for } \varphi \in [\varphi_{\mathrm{c}}^{-}, \varphi_{\mathrm{c}}^{+}]. \tag{117}$$

As we will see, this is the natural labeling if we want to assign a continuous physical interpretation of each wave between the branching points where the roots change character from being complex to being real.

The amplifications and velocities of these waves are plotted in Figure 2 as functions of χ as given by (106). Note in particular that the frozen-like and equilibrium-like mass fraction waves Y_{f} and Y_{e} *overlap* in the critical region; herein, they are truly separate waves. They are "backwardly connected" by the indeterminate wave X, which has no physical interpretation in terms of the waves existing in the frozen and equilibrium limit systems. Note that the wave X also serves to connect the frozen-like and equilibrium-like sonic waves.

7. Summary

We have investigated the wave dynamics of a two-phase flow model with relaxation towards phase equilibrium. We have shown that for *any* given thermodynamic substance, physical state, relaxation rate and chosen wave number k, the velocities and amplifications of the resulting plane waves are determined by only two dimensionless numbers (denoted as r and φ in our paper).

We have provided a complete description of the wave dynamics in terms of these two numbers. In particular, we have stated general closed-form expressions for the corresponding wave velocities and amplifications. Our results hold generally for any such relaxation model where the relaxation parameter ε is differentiable across the equilibrium state. This complete description entails the following specific, generally valid results:

- The wave velocities and amplification factors possess Galilean symmetry;
- The stability criterion for the transitional waves is precisely the classical *subcharacteristic condition*;
- The velocities of the transitional waves satisfy a natural *maximum principle*, related to causality: The maximum transitional wave speed can never exceed the maximum frozen wave speed.

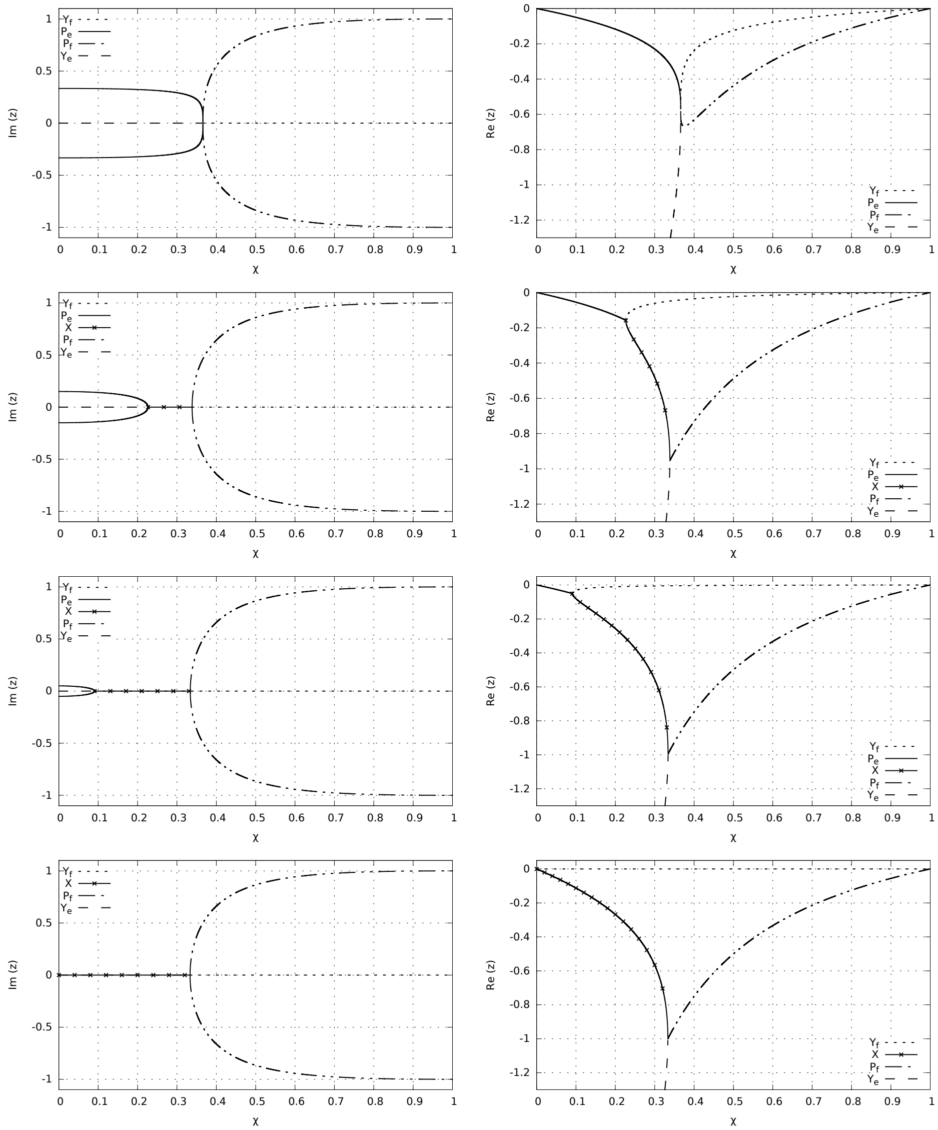

FIGURE 2. Transitional wave properties. Left: Im(z) (velocities). Right: Re(z) (amplifications). Top to bottom: $r = 1/3$, $r = 0.15$, $r = 0.05$, $r = 0$.

Of particular interest is the emergence of a *critical region* defining a non-smooth transition between the equilibrium and frozen limits. This extends the previous similar observation of Aursand and Flåtten[3] for 2×2 systems.

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