WAVE DYNAMICS OF LINEAR HYPERBOLIC RELAXATION SYSTEMS

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Abstract. We consider linear hyperbolic systems with a stable rank 1 relaxation term. We establish that the characteristic polynomial for the individual Fourier components of the solution can be written as a convex combination of the eigenvalue polynomials for the formal stiff and non-stiff limits. This allows us to provide a direct and elementary proof of the equivalence between linear stability and the subcharacteristic condition. In a similar vein, a maximum principle follows: the velocity of each individual Fourier component is bounded by the minimum and maximum eigenvalues of the non-stiff limit system.

1. Introduction

We are interested in hyperbolic conservation laws with relaxation source terms, acting to drive the system towards an equilibrium state. Such systems have many applications in the modeling of natural phenomena [2, 7], in particular they are useful for describing the interaction between fluid-mechanical and thermodynamical processes [9, 16, 17].

In one space dimension, hyperbolic relaxation systems can be written in the general form

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \frac{1}{\varepsilon}\boldsymbol{Q}(\boldsymbol{U}), \tag{1}$$

to be solved for the vector $\boldsymbol{U}(x,t) \in G \subseteq \mathbb{R}^N$. Herein, $\boldsymbol{F}(\boldsymbol{U})$ is the flux vector and $\boldsymbol{Q}(\boldsymbol{U})$ is the relaxation term. The parameter $\varepsilon > 0$ determines the rate of relaxation towards equilibrium. The system is hyperbolic when $\boldsymbol{F_U}(\boldsymbol{U})$ has real eigenvalues and is diagonalizable and strictly hyperbolic when all the eigenvalues are real and distinct.

Two limit cases of (1) are of particular interest and will be central to the investigations of this paper:

- The *non-stiff limit*, characterized by $\varepsilon \to \infty$. In this limit, we may write (1) as

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = 0. \tag{2}$$

  We will denote (2) as the *homogeneous system.*
- The formal *equilibrium limit*, characterized by $\boldsymbol{Q}(\boldsymbol{U}) \equiv 0$. This assumption defines an equilibrium manifold [5] through

$$M = \{\boldsymbol{U} \in G : \boldsymbol{Q}(\boldsymbol{U}) = 0\}. \tag{3}$$

  Imposing local equilibrium, we may express (1) as

$$\partial_t \boldsymbol{u} + \partial_x \boldsymbol{f}(\boldsymbol{u}) = 0, \tag{4}$$

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for some reduced variable $\boldsymbol{u}(x,t) \in \mathbb{R}^n$, where $n \leq N$. Herein, every $\boldsymbol{u}$ uniquely defines an equilibrium state $\mathcal{E}(\boldsymbol{u}) \in M$.

We will denote (4) as the *equilibrium system.*

A highly relevant question is whether solutions to (1) converge to solutions to (4) as $\varepsilon \to 0$. Chen [4] gives an overview of the existing literature where some important results are included. For the solution of a relaxation system to have a well-behaved limit, stability of the solution is a necessary criterion. Chen et al. [5] introduce an entropy condition which ensures dissipativity of the first-order Chapman-Enskog type expansion. If the full system is endowed with such an entropy, there will also exist a strictly convex entropy for the equilibrium system, implying that the equilibrium system will be hyperbolic. Further, under some suitable assumptions, it has been proved [5, 6] that the solution of the 2×2 relaxation system strongly converges to that of the equilibrium system. Lattanzio and Marcati [11] prove convergence for the same system by using a compensated compactness argument. Such arguments were first developed in this context by Chen and Liu [6]. Yong [22, 23] establishes a *relaxation criterion* which is necessary for the convergence of solutions as $\varepsilon \to 0$ in the non-linear case. For strictly non-linear systems, stronger stability assumptions, including the existence of a strictly convex entropy, are needed [22]. Lorenz and Schroll [14] prove equivalence between the relaxation criterion and the convergence of the solution as $\varepsilon \to 0$ in L^2 for linear systems with constant coefficients. It is also possible to show that existence of a strictly convex entropy function is not needed for linear systems to have a well-behaved limit [15, 22]. Tzavaras [20] builds a framework for using the zero relaxation limit to approximate hyperbolic systems of conservation laws when the solutions of the limiting systems are assumed to be smooth.

1.1. **The subcharacteristic condition.** A key concept arising in the analysis of hyperbolic relaxation systems is the *subcharacteristic condition* [12, 21], first introduced by Leray and subsequently independently found by Whitham. The modern terminology was introduced by Liu [13] for 2×2 systems.

For general $N\times N$ hyperbolic systems, the condition may be stated as follows [5].

Definition 1. *Let the N eigenvalues of the homogeneous system* (2) *be given by*

$$\lambda_1 \leq \cdots \leq \lambda_k \leq \lambda_{k+1} \leq \cdots \leq \lambda_N, \tag{5}$$

i.e. λ_k are the eigenvalues of

$$\boldsymbol{A} = \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{U}}. \tag{6}$$

Let $\tilde{\lambda}_j$ be the n eigenvalues of the equilibrium system (4)*, i.e. $\tilde{\lambda}_j$ are the eigenvalues of*

$$\boldsymbol{B} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}. \tag{7}$$

Herein, the homogeneous system (2) *is applied to a local equilibrium state $\boldsymbol{U} = \mathcal{E}(\boldsymbol{u})$, such that*

$$\lambda_k = \lambda_k(\mathcal{E}(\boldsymbol{u})), \qquad \tilde{\lambda}_j = \tilde{\lambda}_j(\boldsymbol{u}). \tag{8}$$

The equilibrium system (4) *is said to satisfy the* **subcharacteristic condition** *with respect to* (2) *when the following statements hold:*

(1) *all $\tilde{\lambda}_j$ are real;*

(2) *if the* $\tilde{\lambda}_j$ *are sorted in ascending order as*

$$\tilde{\lambda}_1 \leq \cdots \leq \tilde{\lambda}_j \leq \tilde{\lambda}_{j+1} \leq \cdots \leq \tilde{\lambda}_n, \tag{9}$$

then $\tilde{\lambda}_j$ *are interlaced with* $\tilde{\lambda}_k$ *in the following sense: Each* $\tilde{\lambda}_j$ *lies in the closed interval* $[\lambda_j, \lambda_{j+N-m}]$.

Definition 2. *Assume that the subcharacteristic condition is satisfied in the sense of Definition 1. If in addition each* $\tilde{\lambda}_j$ *lies in the open interval* $(\lambda_j, \lambda_{j+N-m})$*, then the equilibrium system* (4) *is said to satisfy the* **strict subcharacteristic condition** *with respect to* (2).

Chen et al. [5] proved that for general $N \times N$ systems, their entropy condition implies the subcharacteristic condition. Yong [22] proved that for relaxation systems satisfying $n = N-1$, the subcharacteristic condition is necessary for the linear stability of the equilibrium state; hence it is also necessary for convergence.

For non-linear 2×2 systems, it has been well established that stability of the equilibrium state is equivalent to the strict subcharacteristic condition [5, 23]. In particular, Chen et al. [5] showed that the strict subcharacteristic condition is equivalent to their entropy condition in this case.

1.2. **Contributions of the current paper.** This paper is motivated by the observation that the modern mathematical literature focuses strongly on nonlinear analytical techniques, with the aim of obtaining asymptotic behavior and zero relaxation limits for nonlinear systems. However, there is also much physical insights to be gained through simple linear analyses on systems in the form (1). By linearizing the system and applying a von Neumann type analysis, one obtains dispersion relations giving the amplifications and velocities of individual Fourier components as functions of the wave number. Such analyses have been performed on two-phase flow models by for instance Städtke [19] and Solem et al. [18], and on the St. Venant equations by Barker et al. [3].

Following in the footsteps of Yong [23], a systematic study of the wave dynamics of general 2×2 systems was undertaken in [1]. Here it was established that the velocity of any isolated Fourier component will be a monotonic function of the wave number or the relaxation time. A critical phenomenon was also observed; if the eigenvalues of the homogeneous system are symmetric around the eigenvalue of the equilibrium system, a definite branching point in the wave number can be identified. At this wave number, both velocities and amplification factors are equal and non-differentiable.

Similar critical phenomena were observed for the two-phase flow model investigated in [18]. In particular, it was observed that if the ratio of the equilibrium and homogenous sound speeds is less than 1/3, the characteristics of the sound waves cannot be continously connected between the homogeneous and equilibrium limits as functions of the wave number or relaxation time; there exist concrete *transition points* where the system changes character in a very qualitative manner.

In this paper, we expand on the works [1, 18] by considering general linear $N \times N$ systems with a stable relaxation operator of rank 1, i.e. $n = N-1$. For this case, we prove a useful proposition:

- P1: *The characteristic polynomial for any isolated Fourier component can be written as a convex combination of the limiting homogeneous and equilibrium eigenvalue polynomials.*

This result, following from elementary linear algebra and consistent with the observation made in [18], allows for obtaining dispersion relations for any such rank 1 hyperbolic relaxation system directly from the homogeneous and equilibrium eigenvalues; no explicit knowledge of the detailed structure of the relaxation operator is needed. Hence this proposition provides a tool for significantly simplifying the kind of von Neumann type analysis as performed in [1, 18, 19].

This proposition also provides an heuristic benefit in describing the fundamental relationship between stability, causality and the subcharacteristic condition. In particular, by using basic properties of polynomials established in the modern literature [8, 24], we are able to provide direct and elementary proofs of the following expected results:

P2: *For strictly hyperbolic systems with a stable rank 1 relaxation term, the linear stability condition is precisely the subcharacteristic condition.*

P3: *If the subcharacteristic condition holds for such systems, a maximum principle follows: the velocity of any isolated Fourier component is bounded by the maximum and minimum eigenvalues of the homogeneous system.*

These propositions will be precisely formulated in the main part of our paper, which is organized as follows. In Section 2, we obtain the linearized system around the equilibrium state. In Section 2.2, we derive the characteristic polynomial for a Fourier component of wave number k and prove Proposition P1. In Section 3, we provide an elementary proof of Proposition P2; the equivalence between linear stability and the subcharacteristic condition. In Section 4, we provide an elementary proof of Proposition P3, which has the interpretation as a causality principle.

Finally, in Section 5, the results of our paper are summarized.

2. Linearized relaxation systems

Henceforth, we will consider linearized relaxation systems. Let $\boldsymbol{U}_{\text{eq}} \in G$ be an equilibrium state, i.e. a constant N-vector characterized by $\boldsymbol{Q}(\boldsymbol{U}_{\text{eq}}) = 0$. The relaxation system (1) linearized around $\boldsymbol{U}_{\text{eq}}$ can then be written as

$$\partial_t \boldsymbol{V} + \boldsymbol{A}\, \partial_x \boldsymbol{V} = \frac{1}{\varepsilon} \boldsymbol{R}\boldsymbol{V}, \tag{10}$$

where $\boldsymbol{V} = \boldsymbol{U} - \boldsymbol{U}_{\text{eq}}$. Herein

$$\boldsymbol{A} = \left.\frac{\partial \boldsymbol{F}(\boldsymbol{U})}{\partial \boldsymbol{U}}\right|_{\boldsymbol{U}_{\text{eq}}} \quad \text{and} \quad \boldsymbol{R} = \left.\frac{\partial \boldsymbol{Q}(\boldsymbol{U})}{\partial \boldsymbol{U}}\right|_{\boldsymbol{U}_{\text{eq}}} \tag{11}$$

are both $N \times N$ matrices with constant coefficients.

2.1. Plane-wave solutions.

For the purpose of the present analysis, we write the solution to the linearized problem (10) in terms of its Fourier components. Following the approach of Yong [22, 23], we assume initial data $\boldsymbol{V}(x, 0) \in L^2([a, b])$, where $[a, b] \subset \mathbb{R}$ is some interval, and write the unique solution to the linear initial value problem as

$$\boldsymbol{V}(x, t) = \sum_k \boldsymbol{V}_k(x, t) = \sum_k \exp(\boldsymbol{H}(k)t) \exp(ikx) \hat{\boldsymbol{V}}(k) \tag{12}$$

where

$$\boldsymbol{H}(k) = \frac{1}{\varepsilon} \boldsymbol{R} - ik\boldsymbol{A}. \tag{13}$$

Furthermore, we can write $\boldsymbol{H} = \boldsymbol{P}\boldsymbol{J}\boldsymbol{P}^{-1}$, where $\boldsymbol{P}$ is the matrix of generalized eigenvectors and $\boldsymbol{J}$ is the corresponding Jordan matrix. Now let λ_j denote the eigenvalues of $\boldsymbol{H}$. The solution (12) can then be written as a combination of elementary waves as

$$\boldsymbol{V}(x,t) = \sum_k \sum_{j=1}^{N} \tilde{V}_j(k,t)\exp\left(ikx + \lambda_j t\right), \tag{14}$$

for some amplitudes $\tilde{V}_j(k,t)$, which depend on k and are polynomials in t. Notice that there is a plane wave solution associated with each distinct eigenvalue. In particular, if $\boldsymbol{H}$ is diagonalizable, $\boldsymbol{J}$ reduces to the diagonal matrix consisting of the eigenvalues of $\boldsymbol{H}$, and $\tilde{V}_j(k,t) = \tilde{V}_j(k)$ for all j.

Considering (14), it is natural to introduce the dispersion relation

$$v_j(k) = -\frac{1}{k}\mathrm{Im}(\lambda_j) \tag{15}$$

and the dampening

$$f_j(k) = \mathrm{Re}(\lambda_j) \tag{16}$$

in order to further re-write the solution as

$$\boldsymbol{V}(x,t) = \sum_k \sum_{j=1}^{N} \tilde{V}_j(k,t)\exp(f_j(k))\exp\left(ik(x - v_j(k)t)\right). \tag{17}$$

This allows us to describe the full wave dynamics of the linear relaxation system (10) in terms of the eigenvalues of the matrix $\boldsymbol{H}$. Note also that since we have the symmetry

$$\boldsymbol{H}(k) = \overline{\boldsymbol{H}(-k)}, \tag{18}$$

we can study the system for wave numbers $k \in [0,\infty)$ without loss of generality.

2.1.1. *Stability.* We say that the relaxation system (1) is *linearly stable* if the solutions (12) to its linearization (10) around the equilibrium state $\boldsymbol{U}_{\mathrm{eq}}$ are bounded in L^2 for all $t \in [0,\infty)$. This is equivalent to the condition

$$|\exp(\boldsymbol{H}(k)t)| \le C \quad \forall k \in \mathbb{R}, \tag{19}$$

where C is some positive constant and $|\cdot|$ denotes the L^2 norm for matrices. By making the variable transformations

$$\eta = \frac{t}{\varepsilon}, \qquad \xi = -kt \tag{20}$$

we may state the stability condition (19) in the following form:

Definition 3. *Consider the relaxation system* (1) *linearized as* (10) *around the state* $\boldsymbol{U}_{\mathrm{eq}}$. *Assume that there is a* $C > 0$ *such that*

$$|\exp\left(\eta R + i\xi A\right)| \le C \tag{21}$$

for all $\eta \ge 0$ *and* $\xi \in \mathbb{R}$.

Then the equilibrium state $\boldsymbol{U}_{\mathrm{eq}}$ *is said to be* **linearly stable**.

This is precisely the *stability criterion* identified by Yong [22], as part of his stronger *relaxation criterion.*

We may now state the following Lemma [10].

Lemma 1. *Linear stability in the sense of Definition 3 is equivalent to the following statements being valid for all k:*

- *All eigenvalues λ_j of the matrix $\boldsymbol{H}(k)$ have a real part $\mathrm{Re}(\lambda_j) \leq 0$.*
- *If J_r is a Jordan block of the Jordan matrix $\boldsymbol{J} = \boldsymbol{P}^{-1}\boldsymbol{H}\boldsymbol{P}$ which corresponds to an eigenvalue λ_j with $\mathrm{Re}(\lambda_j) = 0$, then J_r has dimension 1×1.*

Proof. The proof is straightforward and can be found in [10]. □

We also define the stronger notion of *strict stability*:

Definition 4. *Assume that the equilibrium state $\boldsymbol{U}_{\mathrm{eq}}$ is linearly stable in the sense of Definition 3. If in addition all eigenvalues of the matrix $\boldsymbol{H}(k)$ have a real part $\mathrm{Re}(\lambda_j) < 0$ for all k, then the equilibrium state $\boldsymbol{U}_{\mathrm{eq}}$ is said to be* **strictly linearly stable**.

2.2. **The characteristic polynomial.** We assume that the relaxation matrix $\boldsymbol{R}$ is stable, i.e. it has no eigenvalues with positive real parts. For the general linear $N \times N$ system with rank 1 relaxation the matrix $\boldsymbol{H}$ can then, up to a scaling and a similarity transform, be written as

$$\boldsymbol{H} = \frac{1}{\varepsilon}\boldsymbol{R} - ik\boldsymbol{A} = \frac{1}{\varepsilon}\begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ r_{N,1} & \dots & -1 \end{pmatrix} - ik\begin{pmatrix} a_{1,1} & \dots & a_{1,N} \\ \vdots & & \vdots \\ a_{N,1} & \dots & a_{N,N} \end{pmatrix}. \tag{22}$$

A crucial property of the characteristic polynomial of (22) is that it can be written as a convex sum of the polynomials of the homogeneous and equilibrium systems. To obtain this result, we first need to establish the following lemma:

Lemma 2. *Assume that the relaxation matrix is stable. In the context of* (22)*, the characteristic polynomials for the homogeneous system and the equilibrium system are given by*

$$P_h(z) = \det(-i\boldsymbol{A} - z\boldsymbol{I}) \tag{23}$$

and

$$P_e(z) = -\det(-i\boldsymbol{C}\boldsymbol{D}^T - i\boldsymbol{A}_{N,N} - z\boldsymbol{I}), \tag{24}$$

respectively. In the above,

$$\begin{aligned} \boldsymbol{H}_h &= -ik\boldsymbol{A} \\ \boldsymbol{H}_e &= -ik\boldsymbol{B}, \\ z &= \lambda/k \end{aligned}$$

and the vectors $\boldsymbol{D}$, $\boldsymbol{C}$ are given by

$$\boldsymbol{D} = \begin{pmatrix} r_{N,1} \\ \vdots \\ r_{N,N-1} \end{pmatrix} \tag{25}$$

$$\boldsymbol{C} = \begin{pmatrix} a_{1,N} \\ \vdots \\ a_{N-1,N} \end{pmatrix} \tag{26}$$

and

$$\boldsymbol{A}_{N,N} = \begin{pmatrix} a_{1,1} & \cdots & a_{1,N-1} \\ \vdots & & \vdots \\ a_{N-1,1} & \cdots & a_{N-1,N-1} \end{pmatrix}. \tag{27}$$

Proof. It is easily seen that (23) is the characteristic polynomial for the homogeneous system. To see that the characteristic polynomial for the equilibrium system satisfies (24), we look at solutions $\boldsymbol{V}$ satisfying $\boldsymbol{R}\boldsymbol{V} = 0$. With $\boldsymbol{R}$ as in (22) and $\boldsymbol{V} = [V_1, V_2, \ldots, V_N]^T$, we have

$$\sum_{k=1}^{N-1} r_{Nk} V_k - V_N = 0, \tag{28}$$

such that the equilibrium system with $\boldsymbol{v} = [V_1, \ldots, V_{N-1}]$ is equal to

$$\partial_t \boldsymbol{v} + \begin{pmatrix} a_{11} & \cdots & a_{1,N-1} \\ \vdots & & \vdots \\ a_{N-1,1} & \cdots & a_{N-1,N-1} \end{pmatrix} + \begin{pmatrix} a_{1,N} \\ \vdots \\ a_{N-1,N} \end{pmatrix} \begin{pmatrix} r_{N,1} \cdots r_{N,N-1} \end{pmatrix} \partial_x \boldsymbol{v} = 0 \tag{29}$$

Thus, the equilibrium system has the characteristic equation $\det(-i\boldsymbol{B} - z\boldsymbol{I}) = 0$, with

$$\boldsymbol{B} = \boldsymbol{C}\boldsymbol{D}^T + \boldsymbol{A}_{N,N}. \tag{30}$$

□

We can now establish the following:

Proposition 1. *Assume that the relaxation matrix is stable. Let*

$$\chi = \frac{\varphi}{\varphi + 1} \tag{31}$$

with $\varphi = k\varepsilon$. The characteristic polynomial for (22) can be written in the form

$$\Psi(z) = \chi P_h(z) + (1 - \chi) P_e(z) = 0, \quad \chi \in [0, 1], \tag{32}$$

where $P_h(z)$ and $P_e(z)$ are given by (23) and (24), respectively.

Proof. We have

$$\boldsymbol{H} - \lambda \boldsymbol{I} = \frac{1}{\varepsilon}\boldsymbol{R} - ik\boldsymbol{A} - \lambda \boldsymbol{I} = \begin{pmatrix} -ika_{11} - \lambda & \cdots & -ika_{1N} \\ \vdots & & \vdots \\ \frac{r_{N1}}{\varepsilon} - ika_{N1} & \cdots & \frac{-1}{\varepsilon} - ika_{NN} - \lambda \end{pmatrix}. \tag{33}$$

Multiplying the characteristic equation of $\boldsymbol{H}$ with k^n, we get

$$\det\left(\frac{1}{\varphi}\boldsymbol{R} - i\boldsymbol{A} - z\boldsymbol{I}\right) = \det \begin{pmatrix} -ia_{11} - z & \cdots & -ia_{1N} \\ \vdots & & \vdots \\ \frac{r_{N1}}{\varphi} - ia_{N1} & \cdots & \frac{-1}{\varphi} - ia_{NN} - z \end{pmatrix} = 0, \tag{34}$$

where $\varphi = k\varepsilon$ and $z = \lambda/k$. Introducing $\boldsymbol{A}_{n,k}$ as the sub-matrix of $-i\boldsymbol{A} - z\boldsymbol{I}$ where the nth row and the kth column is removed, we have the characteristic equation in the following form,

$$\tilde{\Psi}(z) = \sum_{k=1}^{N-1} (-1)^{k-1} r_{Nk} \cdot \det(\boldsymbol{A}_{N,k}) - \det(\boldsymbol{A}_{N,N}) + \varphi \cdot \det(-i\boldsymbol{A} - z\boldsymbol{I}) = 0 \tag{35}$$

when expanding along the bottom row of (34). By (23), we may write (35) as

$$\tilde{\Psi}(z) = \tilde{P}_e(z) + \varphi P_h(z), \tag{36}$$

where

$$\tilde{P}_e(z) = \sum_{k=1}^{N-1} (-1)^{k-1} r_{Nk} \cdot \det(\boldsymbol{A}_{N,k}) - \det(\boldsymbol{A}_{N,N}). \tag{37}$$

We can now observe that (37) is equal to the characteristic polynomial for the equilibrium system (24) by the following calculation,

(38)

$$\begin{aligned}
\tilde{P}_e(z) =& \det \begin{pmatrix} -ia_{11} - z & \cdots & -ia_{1,N-1} & -ia_{1N} \\ \vdots & \ddots & & \vdots \\ -ia_{N-1,1} & \cdots & -ia_{N-1,N-1} - z & -ia_{N-1,N} \\ r_{N,1} & \cdots & r_{N,N-1} & -1 \end{pmatrix} \\
=& \det \begin{pmatrix} -ia_{11} - ia_{1N} r_{N,1} - z & \cdots & -ia_{1,N-1} - ia_{1N} r_{N,N-1} & 0 \\ \vdots & \ddots & & \vdots \\ -ia_{N-1,1} - ia_{N-1,n} r_{N,1} & \cdots & -ia_{N-1,N-1} - ia_{N-1,N} r_{N,N-1} - z & 0 \\ 0 & \cdots & 0 & -1 \end{pmatrix} \\
=& -\det(-i\boldsymbol{B} - z\boldsymbol{I}) = P_e(z),
\end{aligned}$$

where we have first added $-ia_{jN}$ multiplied with the last row to the jth row of (38) and then added $r_{N,i}$ multiplied with the last column to the jth column for $j = 1, \ldots, N-1$. Substituting (31) into (36) we obtain

$$\Psi(z) = (1-\chi)\tilde{\Psi}(z) = \chi P_h(z) + (1-\chi) P_e(z). \tag{39}$$

□

3. Linear stability

In this section we prove that strictly hyperbolic relaxation systems with stable rank 1 relaxation matrices are linearly stable if and only if the roots of the two limiting polynomials interlace on the imaginary axis, i.e. if and only if the relaxation system satisfies the subcharacteristic condition of Definition 1.

Let

$$\Psi(z) = \chi P_h(z) + (1-\chi) P_e(z) \tag{40}$$

be the eigenvalue polynomial for the $N \times N$ linear hyperbolic relaxation system (10) where $\boldsymbol{R}$ is of rank one and stable. Further, let $P_h(z)$ and $P_e(z)$ be as in Lemma 2. Since $\boldsymbol{A}$ is a $N \times N$ real matrix and $\boldsymbol{B}$ a $(N-1) \times (N-1)$ real matrix, the coefficients of $P_h(z)$ and $P_e(z)$ alternate between being purely real and purely imaginary in the following way:

$$P_h(z) = z^N + ib_{N-1}z^{N-1} + b_{N-2}z^{N-2} + \dots \tag{41}$$

$$P_e(z) = z^{N-1} + ic_{N-2}z^{N-2} + c_{N-3}z^{N-3} + \dots, \tag{42}$$

such that the full polynomial (40) satisfies

$$\begin{aligned}\Psi(z) =& \chi P_h(z) + (1-\chi)P_e(z) \\ =& \chi(z^N + ib_{N-1}z^{N-1} + b_{N-2}z^{N-2} + \dots) \\ &+ (1-\chi)(z^{N-1} + ic_{N-2}z^{N-2} + c_{N-3}z^{N-3} + \dots).\end{aligned} \tag{43}$$

By rewriting the polynomial in this form, we are able to prove the following proposition.

Proposition 2. *Assume that* $\boldsymbol{R}$ *is a stable rank 1 relaxation matrix for the linearized relaxation system* (10) *Let* $\Psi(z)$ *in* (40) *be its characteristic polynomial. Further, assume that the system is strictly hyperbolic. Then the system* (1) *is linearly stable for all* $\chi \in [0,1]$ *if and only if the roots of* $P_e(z)$ *are purely imaginary and interlace the roots of* $P_h(z)$ *on the imaginary axis, i.e. the subcharacteristic condition is satisfied.*

Further, the subcharacteristic condition is strictly satisfied if and only if the system is strictly linearly stable for all $\chi \in (0,1)$.

Before we prove Proposition 2 for relaxation systems, let us take a look at a general complex polynomial

$$P(z) = \sum_{k=0}^{N}(a_k + ib_k)z^k, \quad a_N + ib_N \neq 0. \tag{44}$$

This polynomial can be rewritten as $P(z) = m(z) + p(z)$, where $m(z)$ and $p(z)$ are the two axially complementary polynomials

$$m(z) = \frac{1}{2}\left[P(z) + \overline{P(-\overline{z})}\right] \tag{45}$$

$$p(z) = \frac{1}{2}\left[P(z) - \overline{P(-\overline{z})}\right]. \tag{46}$$

Let us assume that $m(z)$ and $p(z)$ have no roots in common. The order of $m(z)$ is one more than the order of $p(z)$ if N is even and the opposite if N is odd. Further, observe that the inequality

$$a_N a_{N-1} + b_N b_{N-1} > 0 \tag{47}$$

is necessary for (44) to be stable, as the coefficient for z^{n-1} is equal to minus the sum of all the roots and that the real part of the sum of the roots have to be less than zero. The following stability lemma exists for general polynomials [24].

Lemma 3. *Assume that* (47) *holds and that* $m(z)$ *and* $p(z)$ *have no roots in common. Then the general complex polynomial* (44) *is strictly stable, i. e.* $\mathrm{Re}(\lambda) < 0$ *for all roots* λ*, if and only if* $m(z)$ *and* $p(z)$ *have distinct purely imaginary roots that interlace on the imaginary axis.*

Proof. The proof is presented in Zhareddine [24]. □

Now we are ready to prove Proposition 2.

Proof. We assume that all roots $P_h(z)$ and $P_e(z)$ have in common have been factored out such that we are left with the reduced polynomial $\Psi_r(z)$. All the roots that $P_{e,r}$ and $P_{h,r}$ have in common satisfy $\mathrm{Re}(\lambda) = 0$. Since the system is assumed to be strictly hyperbolic, all the roots that $P_e(z)$ and $P_h(z)$ have in common are distinct. Thus, the Jordan blocks corresponding to these eigenvalues will have dimension 1×1 and, according to Lemma 1, they will not cause any linear instability in the sense of Definition 3.

We now split the remaining polynomial $\Psi_r(z)$ into two axially complementary polynomials as in (45),

$$m(z) = \frac{1}{2}\left[\Psi_r(z) + \overline{\Psi_r(-\overline{z})}\right] \tag{48}$$

$$p(z) = \frac{1}{2}\left[\Psi_r(z) - \overline{\Psi_r(-\overline{z})}\right]. \tag{49}$$

We observe that $m(z) = \chi P_{h,r}(z)$ and $p(z) = (1-\chi)P_{e,r}(z)$ if N is even and $p(z) = \chi P_{h,r}(z)$ and $m(z) = (1-\chi)P_{e,r}(z)$ if N is odd, making $P_h(z)$ and $P_e(z)$ axially complementary. We easily see that for $\chi = 1$ we have the homogeneous eigenvalue polynomial and for $\chi = 0$ we have the equilibrium eigenvalue polynomial.

From now on, we look at $\chi \in (0, 1)$. Corresponding to the coefficients for the general polynomial (44), $\Psi_r(z)$ has

$$a_N = \chi, \quad b_N = 0, \quad a_{N-1} = (1-\chi), \tag{50}$$

such that (47) always is fulfilled. It now follows from Lemma 3 that the roots $\{z_j\}$ of $\Psi_r(z)$ satisfy $\mathrm{Re}(z_j) < 0$ if and only if the roots of $P_{h,r}(z)$ interlace the roots of $P_{e,r}$ and their roots are distinct and purely imaginary.

If $P_e(z)$ and $P_h(z)$ have no roots in common, we have $\mathrm{Re}(z_j) < 0$ for all roots of $\Psi(z)$ when $\chi \in (0, 1)$, making the system strictly linearly stable.

If all roots of $P_e(z)$ are roots of $P_h(z)$, the remaining eigenvalue polynomial will have one root, $\Psi_r(z) = \varphi(z - z_k) + (\varphi - 1)$, where z_k is a root of $P_h(z)$. This root is always stable as $\varphi \leq 1$. □

With Proposition 2, we have now shown that there is an equivalence between linear stability and the subcharacteristic condition for hyperbolic relaxation systems with stable rank one relaxation matrices. We can further observe that linear stability implies that the linear equilibrium system must be strictly hyperbolic.

4. A maximum principle

We show that the velocities of the linearized hyperbolic relaxation system 10 can never exceed the velocities of the corresponding homogeneous system when the system is linearly stable. We prove this with the help of some properties for polynomials found in Fisk [8].

Let (40) be the eigenvalue polynomial for the strictly hyperbolic $N \times N$ linearized relaxation system with a relaxation matrix of rank 1. Assume that the system is linearly stable. Let $\Psi_r(z)$ be the reduced polynomial where all the roots that $P_e(z)$ and $P_h(z)$ have in common are factored out. Then, by Proposition 2, the roots of $P_{h,r}(z)$ strictly interlace the roots of $P_{e,r}(z)$ on the imaginary axis. We make a translation of the roots from the left half plane to the lower half plane,

$$\begin{aligned}\hat{\Psi}_r(z) &= i^N \Psi_r(-iz) \\ &= i^N \chi P_{h,r}(-iz) + i^N (1-\chi) P_{e,r}(-iz) \\ &= h(z) + i g(z)\end{aligned} \tag{51}$$

The roots of $h(z)$ and $g(z)$ in (51) interlace on the real axis. Further, the real roots of $h(z)$ and $g(z)$ correspond to the roots of $P_{h,r}(z)$ and $P_{e,r}(z)$ on the imaginary axis. Since the roots $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ of $h(z)$ strictly interlace the roots $\{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}\}$ of $g(z)$, $g(z)$ changes sign on the roots of $h(z)$.

$h(z)$ is of order N and of one order more than $g(z)$. The homogeneous system is assumed to be strictly hyperbolic, making the roots of $h(z)$ distinct such that

$$\frac{h(z)}{z-\lambda_1}, \quad \frac{h(z)}{z-\lambda_2}, \ldots, \frac{h(z)}{z-\lambda_N} \tag{52}$$

is a basis for all real polynomials with real roots of order $N-1$. We can therefore express $g(z)$ with basis

$$g(z) = \sum_{k=1}^{N} c_k \frac{h(z)}{z-\lambda_k}. \tag{53}$$

The c_ks have the same sign if the eigenvalue polynomial (40) is strictly stable. For a root λ_k of $h(z)$, we have

$$g(\lambda_k) = c_k(\lambda_k - \lambda_1)\ldots(\lambda_k - \lambda_{k-1})(\lambda_k - \lambda_{k+1})\ldots(\lambda_k - \lambda_N), \tag{54}$$

such that

$$\operatorname{sgn}(g(\lambda_k)) = \operatorname{sgn}(c_k)(-1)^{k+N} \tag{55}$$

and we can see that $g(z)$ changes sign on the roots of $h(z)$ if all the c_ks have the same sign. The c_ks also have to be strictly greater than zero. If not, λ_k would also be a root of $g(z)$, which contradicts the fact that $h(z)$ and $g(z)$ have no roots in common. P_e in (40) has a positive leading coefficient which makes all the c_ks positive.

We can now prove the following proposition.

Proposition 3. *Let* (10) *be a strictly hyperbolic relaxation system with a stable rank 1 relaxation matrix. Let the system be stable. Then the imaginary parts of the roots,* $\mathrm{Im}(z_k)$*, for* $k = 1, \ldots, N$*, of* (40) *satisfies*

$$\min_i\{\lambda_i)\} \leq \mathrm{Im}(z_k) \leq \max_i\{\lambda_i\} \tag{56}$$

where $i\lambda_k$ *are the roots of* $P_h(z)$*, for all* $\chi \in [0,1]$.

Proof. When the system is stable, the reduced polynomial of (40) is strictly stable for all $\chi \in (0,1)$, the roots of $\chi P_{h,r}(z)$ strictly interlace the roots of $(1-\chi)P_{e,r}(z)$ on the imaginary axis. We look at the translated polynomial in (51). We can write (51) as

$$\hat{\Psi}_r(z) = h(z) + i \sum_{k=1}^{N} c_k \frac{h(z)}{z-\lambda_k}. \tag{57}$$

For a root z_i of (51), we will have

$$0 = 1 + i\sum_{k=1}^{N} c_k \frac{1}{z_i - \lambda_k}. \tag{58}$$

Assume that z_i is a root of $\hat{\Psi}_r(z)$ with $\mathrm{Re}(z_i) > \lambda_k$ for all $k = 1, \dots, N$. All the c_ks are greater than zero, making the real part of the sum in (58) greater than zero, such that the right hand side cannot be equal to zero. Therefore, there are no roots z_i of (40) with real part greater than all the roots of $h(z)$. The proof for $\mathrm{Re}(z_i) < \lambda_k$ is similar.

We conclude that (51) has no roots with real part greater than or smaller than the real roots of $h(z)$. Translating (51) back to (40), we observe that the real part of the roots in (51) correspond to the imaginary parts of the roots in (40).

The roots that $P_e(z)$ and $P_h(z)$ have in common are constant for any $\chi \in [0, 1]$ and will never be able to exceed any maximum or minimum value. □

Remark 1. *The converse direction of Proposition 3 does not hold. We can easily generate two polynomials $P_1(z)$ and $P_2(z)$ satisfying the maximum principle that do not interlace, making the convex combination unstable,*

$$P_1(z) = (z + i5)(z + i)(z - i2) \tag{59}$$

$$P_2(z) = (z + i4)(z + i2). \tag{60}$$

5. Summary

We have provided some fundamental and elementary results pertaining to the von Neumann type analysis of linearized hyperbolic relaxation systems where the relaxation operator is assumed to be stable and of rank 1. Our results may be briefly summarized as follows:

- P1: The characteristic polynomial for any Fourier component of wave number k may be directly obtained as a convex combination of the eigenvalue polynomials for the homogeneous and equilibrium limits.
- P2: A strictly hyperbolic relaxation system with a stable rank 1 relaxation operator is linearly stable if and only if the subcharacteristic condition is satisfied.
- P3: If the subcharacteristic condition is satisfied, the velocity of any isolated Fourier component is bounded by the maximum and minimum eigenvalue of the homogeneous system.

Herein, it should be noted that the proof of P1 is obtained from elementary linear algebra and the statements of P2 and P3 are unsurprising given the already established strong relationship between stability and the subcharacteristic condition [5, 23]. In our opinion, the main interest of our paper lies in the connection provided between theory describing general properties of roots of polynomials and fundamental causality and stability properties of hyperbolic relaxation systems. These connections seem so far to have been given little emphasis in the literature.

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